

Linear Transformations and Functions of Positive Type.

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In a communication received by the Society on April 12 last* I gave in outline a proof that a symmetrisable function $\kappa(s, t)$, which belongs to one of certain classes, admits an expansion in terms of its bi-orthogonal fundamental functions. Two classes were explicitly mentioned, viz., (i) the functions defined by $\kappa(s, t) = \alpha(s) \gamma(s, t)$, and (ii) those defined by

$$\kappa(s, t) = \int_a^b \alpha(s, x) \gamma(x, t) dx.$$

It was assumed in each case that $\gamma(s, t)$ was of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$; in the first case, that $\alpha(s)$ was a function defined in the interval (a, b) ; and, in the second case, that $\alpha(s, t)$ was a symmetric function defined in the square $a \leq s \leq b$, $a \leq t \leq b$. When $\kappa(s, t)$ belongs to either of the classes mentioned and certain conditions of continuity† are satisfied it was shown that

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s) \psi_n(t)}{\lambda_n} + \sum_{n=1} \frac{\xi_n(s) \eta_n(t)}{\nu_n}, \quad (1)$$

where (i)

$$\begin{aligned} &\phi_1(s), \phi_2(s), \dots, \phi_n(s), \dots \\ &\psi_1(s), \psi_2(s), \dots, \psi_n(s), \dots \end{aligned}$$

is a bi-orthogonal system of fundamental functions for the interval (a, b) , and $\phi_n(s)$, $\psi_n(t)$ satisfy the homogeneous equations

$$\left. \begin{aligned} \phi_n(s) &= \lambda_n \int_a^b \kappa(s, t) \phi_n(t) dt, \\ \psi_n(t) &= \lambda_n \int_a^b \psi_n(s) \kappa(s, t) ds; \end{aligned} \right\} \quad (2)$$

(ii) the functions $\eta_n(s)$ ($n = 1, 2, \dots$) constitute an orthogonal system of functions for the interval (a, b) ; and (iii) each of the pairs of functions $\{\phi_m(s), \eta_n(s)\}$, $\{\xi_m(s), \psi_n(s)\}$, $\{\xi_m(s), \eta_n(s)\}$ is orthogonal for the interval (a, b) , whatever be the values of m and n . The symbols ν_n ($n = 1, 2, \dots$) denote certain positive constants which, by suitable modification of the functions $\xi_n(s)$, may all be taken equal to unity.

* See 'Proceedings,' A, vol. 97, pp. 401-413.

† For $\gamma(s, t)$ the condition was that the function should be continuous with respect to s for each fixed value of t in (a, b) .

It will be evident that a function which belongs to either of the above-mentioned classes is obtained from $\gamma(s, t)$ by a particular kind of linear transformation, *the transformation being with respect to the variable s* . Again, a symmetrisable function such as $\alpha(s)\gamma(s, t)\beta(t)$ is obtained by subjecting $\gamma(s, t)$ to two linear transformations, namely, one with respect to each variable. We are thus led to consider a general class of functions, each of which is obtained by subjecting a function of positive type to linear transformations; and to enquire under what circumstances a function of this class admits expansion in the form (1). The present communication has its origin in an attempt to answer the enquiry just stated.

In the first section a summary of the theory of linear transformations due to F. Riesz* is given, and further developments required for the purpose in view are given. At the commencement of the second section functions of positive type are defined, which are more general than those hitherto considered. The theory of quadratic forms with an infinite number of variables is then applied to show that, when $\kappa(s, t)$ is obtained by subjecting one of the functions of positive type to a symmetric couple of linear transformations, (1) is valid almost everywhere,† provided further series involving null functions are added on the right. In general, the equations (2) are not satisfied at all points of (a, b) , but they are always satisfied *almost everywhere*. It is further shown that, when the function of positive type has the continuity property assumed in the previous communication, and the transformations are of a restricted character, the expansion of $\kappa(s, t)$ is valid everywhere in the square $a \leq s \leq b$, $a \leq t \leq b$. Finally, it appears that, when the transformation with respect to t is still further restricted, the expansion of $\kappa(s, t)$ is of the form (1).

Linear Functional Transformations.

§ 1. Previous to giving the proof of the expansion theorems, it will be convenient to summarise briefly certain definitions and results which belong to the theory of linear functional transformations.

Let F denote the class of real functions which have their squares summable in the interval (a, b) . It is to be observed that this hypothesis does not imply that a function of F has necessarily a single finite value at each point of (a, b) . A function of F may be undefined, or multiple valued, at points of the interval which form a set of zero measure; further, the function may be regarded as having the values $+\infty$ or $-\infty$ at points of a similar set.‡ In

* 'Untersuchungen über Systeme integrierbarer Funktionen Math. Annalen,' vol. 69, pp. 449–497. See particularly §12. The case $p = 2$ only is considered here.

† This statement is correct, but it will be found that the result obtained is of a slightly more general character.

‡ Cf. Riesz, *loc. cit.*, p. 454.

what follows a function will be said to be *well-defined* in (a, b) if it has a single finite value at each point of the interval.

Let Ω be a transformation which converts each member of F into a member of the same class, this member being unique save possibly for an additive null function. If we denote the independent variable by s , and a member of F by $f(s)$, a function which results when the transformation Ω is applied to $f(s)$ is denoted by $\Omega[f(s)]$, or by $\Omega_s[f(s)]$, when it is necessary to indicate the variable with respect to which the transformation is carried out.

A transformation Ω is said to be distributive if, for any two functions $f_1(s), f_2(s)$, which belong to F , and for any two constants, c_1, c_2 , we have

$$\Omega[c_1 f_1(s) + c_2 f_2(s)] = c_1 \Omega[f_1(s)] + c_2 \Omega[f_2(s)], \quad (3)$$

almost everywhere in (a, b) . The transformation is said to be *bounded* if, for any function, $f(s)$, which belongs to F ,

$$\int_a^b \{\Omega[f(s)]\}^2 ds \leq M^2 \int_a^b \{f(s)\}^2 ds,$$

where M is independent of $f(s)$. The lower bound of the values of M , for which the inequality just written is true, may be called the *bound* of the transformation. A transformation which is both distributive and bounded is said to be *linear*.*

As examples of linear transformations we have

$$(i) \quad \Omega[f(s)] = \alpha(s)f(s),$$

where $\alpha(s)$ is bounded and summable in (a, b) , and

$$(ii) \quad \Omega[f(s)] = \int_a^b \alpha(s, x)f(x) dx,$$

where $\alpha(s, t)$ is a function which has its square summable in the region $a \leq s \leq b, a \leq t \leq b$. It will be observed that, in each of the examples cited, the function $\Omega[f(s)]$ may not be defined at points of a set of zero measure. The function, however, belongs to F , as was explained above.

§ 2. A transformation, $\bar{\Omega}$, which is related to Ω in such a way that

$$\int_a^b \Omega[f_1(s)] \cdot f_2(s) ds = \int_a^b f_1(s) \cdot \bar{\Omega}[f_2(s)] ds,$$

for any two functions, $f_1(s), f_2(s)$, which belong to F , is said to be *ad-joint* to Ω . It can be shown that any linear transformation has an adjoint, and that the latter is unique, save for an arbitrary null function.† A transformation, Ω , which is adjoint to itself, is said to be *self-adjoint* or *symmetric*.‡

* Riesz, *loc. cit.*, p. 477.

† Riesz, *loc. cit.*, p. 478.

‡ Riesz, *loc. cit.*, p. 484. In the examples cited above (i) is always symmetric, (ii) is symmetric when $\alpha(s, t)$ is a symmetric function.

In what follows, we shall be concerned with pairs, Ω, Ω' , of transformations, which are such that

$$\int_a^b \Omega[f_1(s)] \Omega'[f_2(s)] ds = \int_a^b \Omega[f_2(s)] \cdot \Omega'[f_1(s)] ds,$$

for every pair of functions, $f_1(s), f_2(s)$, which belong to F. Two transformations related in this manner will be referred to as a *symmetric-couple*.

It will be observed that, if $\mathbf{E}$ is the unit transformation* and Ω is symmetric, then $\mathbf{E}$ and Ω constitute a symmetric couple. More generally, any two symmetric transformations which are permutable form a symmetric couple: an important instance of this is the couple defined by $\Omega[f(s)] = \alpha(s)f(s)$; $\Omega'[f(s)] = \beta(s)f(s)$.† Other examples are easily constructed.

§ 3. Let $f_n(s)$ ($n = 1, 2, \dots$) be a sequence of functions of class F. If a function $f(s)$ of class F can be found such that

$$\lim_{n \rightarrow \infty} \int_a^b \{f(s) - f_n(s)\}^2 ds = 0,$$

the sequence is said to *converge strongly* to $f(s)$.‡ The function, $f(s)$, when it exists, is unique, save for an additive null function.

It is easy to prove that, if $f_n(s)$ ($n = 1, 2, \dots$), $g_n(s)$ ($n = 1, 2, \dots$) are sequences which converge strongly to $f(s)$, $g(s)$ respectively, then

$$\lim_{n \rightarrow \infty} \int_a^b f_n(s) g_n(s) ds = \int_a^b f(s) g(s) ds.$$

From the bounded property, it follows that, if Ω is any linear transformation, the strong convergence of a sequence, $f_n(s)$ ($n = 1, 2, \dots$) to $f(s)$, implies the strong convergence of $\Omega[f_n(s)]$ ($n = 1, 2, \dots$) to $\Omega[f(s)]$.§ The two results stated allow it to be proved that, when Ω and Ω' are any two linear transformations and $f_n(s)$ ($n = 1, 2, \dots$) converges strongly to $f(s)$,

$$\lim_{n \rightarrow \infty} \int_a^b \Omega[f_n(s)] \cdot \Omega'[f_n(s)] ds = \int_a^b \Omega[f(s)] \cdot \Omega'[f(s)] ds.$$

§ 4. Let $f_n(s)$ ($n = 1, 2, \dots$) be a countable set of functions of class F such that the series

$$\int_a^b \{f_1(s)\}^2 ds + \int_a^b \{f_2(s)\}^2 ds + \dots + \int_a^b \{f_n(s)\}^2 ds + \dots, \quad (4)$$

* This transformation is defined by $\mathbf{E}[f(s)] = f(s)$.

† For conditions of summability, see § 1.

‡ Riesz, *loc. cit.*, p. 464; also named 'convergence en moyenné' by E. Fischer.

§ Riesz, *loc. cit.*, p. 464.

is convergent. It follows from a known theorem* that the series

$$\{f_1(s)\}^2 + \{f_2(s)\}^2 + \dots + \{f_n(s)\}^2 + \dots, \quad (5)$$

converges almost everywhere in (a, b) ; that $F(s)$, the sum-function of (5), is summable in (a, b) ; and that $\int_a^b F(s) ds$ is the sum of the series (4).

Let $(x_1, x_2, \dots, x_n, \dots)$ or briefly (x) , be the co-ordinates of a real point in Hilbertian space, and consider

$$f[(x), s] = x_1 f_1(s) + x_2 f_2(s) + \dots + x_n f_n(s) + \dots \quad (6)$$

For any value of s in (a, b) such that $F(s), f_1(s), f_2(s), \dots, f_n(s), \dots$ have all finite values, the right-hand member of (6) is a linear form in Hilbert's sense, (x) being regarded as variable. It follows from what has been said that the member is a linear form for almost all values of s in (a, b) . Again, we have

$$\{f[(x), s]\}^2 \leq F(s) \sum_{n=1}^{\infty} x_n^2.$$

almost everywhere in (a, b) ; hence it is clear that, if (x) is fixed, $f[(x), s]$ belongs to the class F. In view of the two properties stated, we shall refer to

$$x_1 f_1(s) + x_2 f_2(s) + \dots + x_n f_n(s) + \dots, \quad (7)$$

as a linear function associated with the class F, whenever the series (4) is convergent.

We shall say that the linear function (7) associated with the class F is well-defined if (i) the functions $f_n(s)$ are all well-defined and (ii) the series (5) converges at all points of (a, b) . A well-defined linear function is clearly a linear form in Hilbert's sense for each value of s in (a, b) .

§ 5. In virtue of the bounded property of a linear transformation we see that, for any transformation Ω ,

$$\int_a^b \{\Omega[f_1(s)]\}^2 ds + \int_a^b \{\Omega[f_2(s)]\}^2 ds + \dots + \int_a^b \{\Omega[f_n(s)]\}^2 ds \leq M^2 \int_a^b F(s) ds,$$

where M is the bound of the transformation. From this follows the important result that, if (7) is any linear function associated with F, and Ω is any linear transformation, then

$$x_1 \Omega[f_1(s)] + x_2 \Omega[f_2(s)] + \dots + x_n \Omega[f_n(s)] + \dots, \quad (8)$$

is a linear function associated with F.

* A simple proof has been given by Hobson, 'Proc. Lond. Math. Soc.,' Series 2, vol. 8, pp. 27-29. It is easily seen to be unessential to the proof that the functions $s_n(x)$ should be bounded; further, the validity of the theorem is not affected by the hypothesis that each function $s_n(x)$ is either undefined, multiple valued, or infinite at points of a set of measure zero.

Let us denote the sum of the first n terms on the right of (6) by $f_n((x), s)$. Since

$$\int_a^b \{f((x), s) - f_n((x), s)\}^2 ds \leq \sum_{r=n+1}^{\infty} x_r^2 \cdot \int_a^b F(s) ds,$$

it is clear that, when (x) is fixed and n tends to ∞ , $f_n((x), s)$ converges strongly to $f((x), s)$. Hence, as n tends to ∞ , $\Omega[f_n((x), s)]$ converges strongly to $\Omega[f((x), s)]$. But, since (8) is a linear function associated with F , $\Omega[f_n((x), s)]$ converges strongly to $\sum_{n=1}^{\infty} x_n \Omega[f_n(s)]$. We thus see that

$$\Omega[f((x), s)] = \sum_{n=1}^{\infty} x_n \Omega[f_n(s)]$$

almost everywhere in (a, b) at each point (x) of Hilbertian space. Further it will be seen that

$$x_1 \int_a^b \Omega[f_1(s)] ds + x_2 \int_a^b \Omega[f_2(s)] ds + \dots + x_n \int_a^b \Omega[f_n(s)] ds + \dots,$$

is a linear form equal to $\int_a^b \Omega[f((x), s)] ds$.

In passing, we observe that, since $f_n((x), s)$ converges strongly to $f((x), s)$ it follows from the remark at the end of § 3 that

$$\lim_{n \rightarrow \infty} \int_a^b \Omega[f_n((x), s)] \cdot \Omega'[f_n((x), s)] ds = \int_a^b \Omega[f((x), s)] \cdot \Omega'[f((x), s)] ds,$$

where Ω and Ω' are any two linear transformations.

§ 6. We shall say that a linear transformation is *well defined* if, for each well-defined function $f(s)$ of F , the function $\Omega[f(s)]$ is well defined.

Let us suppose that Ω is well-defined, and that the equation (3) holds exactly at *all* points of (a, b) . Then, if $f_1(s), f_2(s), \dots, f_n(s)$ are well-defined functions in (a, b) and $x_1, x_2, \dots, x_n$ are the co-ordinates of a variable point in n dimensional space, we have

$$\begin{aligned} \Omega[x_1 f_1(s) + x_2 f_2(s) + \dots + x_n f_n(s)] \\ = x_1 \Omega[f_1(s)] + x_2 \Omega[f_2(s)] + \dots + x_n \Omega[f_n(s)], \end{aligned}$$

for *all* values of s in the interval (a, b) . This property may be described in the statement that Ω is *precisely distributive in n dimensional space*.

We shall say that a well-defined transformation Ω is *precisely distributive in Hilbertian space* if, corresponding to any well-defined linear function $\sum_{n=1}^{\infty} x_n f_n(s)$, the linear function $\sum_{n=1}^{\infty} x_n \Omega[f_n(s)]$ is well defined and equal to $\Omega[f((x), s)]$, for all points (x) of Hilbertian space and for *all* values of s in (a, b) . It will be seen that, if Ω is *precisely distributive in Hilbertian space*, and if $f_n(s)$ ($n = 1, 2, \dots$) are *well-defined functions such that* $\sum_{n=1}^{\infty} \{f_n(s)\}^2$ *is*

convergent at all points of (a, b) , then the series $\sum_{n=1}^{\infty} \{\Omega[f_n(s)]\}^2$ is convergent at all points of (a, b) .

Let (a', b') be an interval which is part of, or coincident with, (a, b) : and let each end point of (a', b') be regarded as belonging to the interval or not, as may be convenient. Let Ω be a linear transformation which is precisely distributive in Hilbertian space. We shall say that Ω is *perfect* in (a', b') if, corresponding to each function $f(s)$ which is continuous in (a', b') , the function $\Omega[f(s)]$ is continuous in (a', b') .

As an example of the transformations defined, let us consider

$$\Omega[f(s)] = \alpha(s) \cdot f(s).$$

It is easily seen that, if $\alpha(s)$ is well-defined, bounded, and summable in (a, b) , Ω is precisely distributive in Hilbertian space. Further, if $\alpha(s)$ is continuous in an interval (a', b') , Ω is perfect in this interval. As a second example we may take

$$\Omega[f(s)] = \int_a^b \alpha(s, t) f(t) dt.$$

This transformation is precisely distributive in Hilbertian space, if $\{\alpha(s, t)\}^2$ is summable in $a \leq s \leq b$, $a \leq t \leq b$ and $\int_a^b \{\alpha(s, t)\}^2 dt$ exists as a finite number for each value of s in (a, b) .* If we further suppose that $\alpha(s, t)$ is a continuous function in the rectangle $a' \leq s \leq b'$, $a \leq t \leq b$, the transformation is perfect in the closed interval (a', b') . A wider condition than that just stated may, of course, be given.

The Expansion Theorem.

§ 7. The definition of a function of positive type which was given in the previous communication had reference only to bounded functions.† This definition must here be extended.

Let $\gamma(s, t)$ be a symmetric function such that $\{\gamma(s, t)\}^2$ is summable in the square $a \leq s \leq b$, $a \leq t \leq b$. Then, if $f(s)$ is any function which belongs to F, the integral

$$\int_a^b \int_a^b \gamma(s, t) f(s) f(t) ds dt$$

always exists. We shall say that $\gamma(s, t)$ is of *positive type* in the square if the value of the integral is positive or zero for each function $f(s)$ of the class F.

* It may be remarked that this transformation has the property that, corresponding to any (well defined or not) linear function $\sum_{n=1}^{\infty} x_n f_n(s)$ associated with the class F, the linear function $\sum_{n=1}^{\infty} x_n \Omega[f_n(s)]$ is equal to $\Omega[f(\langle x \rangle, s)]$.

† *Loc. cit.*, p. 406.

When the integral is positive for each function $f(s)$ which is not a null function, we say that $\gamma(s, t)$ is *definite*.

Let $\theta_1(s), \theta_2(s), \dots, \theta_n(s), \dots$ be a countable set of functions which belong to F ; and let the series

$$\int_a^b \{\theta_1(s)\}^2 ds + \int_a^b \{\theta_2(s)\}^2 ds + \dots + \int_a^b \{\theta_n(s)\}^2 ds + \dots, \quad (9)$$

be convergent. In view of what was said in § 4 and of the inequality

$$\left\{ \sum_{n=1}^r |\theta_n(s) \theta_n(t)| \right\}^2 \leq \sum_{n=1}^r \left\{ \theta_n(s) \right\}^2 \cdot \sum_{n=1}^r \left\{ \theta_n(t) \right\}^2,$$

it will be clear that the series $\sum_{n=1} \theta_n(s) \theta_n(t)$ is absolutely convergent in the

square $a \leq s \leq b, a \leq t \leq b$, save possibly at such points as belong to a plane set which has for its projection on either axis a set of measure zero. We may express this more briefly in the statement that the series is absolutely convergent *almost everywhere with respect to s , and almost everywhere with respect to t* . This abbreviated form of statement will be employed under similar circumstances throughout the pages which follow.

From the same inequality and the fact that (9) is convergent, it follows easily that

$$\gamma(s, t) = \theta_1(s) \theta_1(t) + \theta_2(s) \theta_2(t) + \dots + \theta_n(s) \theta_n(t) + \dots,$$

has its square summable in $a \leq s \leq b, a \leq t \leq b$. Further, if $f(s)$ is any function of F , it would be found that

$$\int_a^b \int_a^b \gamma(s, t) f(s) f(t) ds dt = \sum_{n=1} \left\{ \int_a^b \theta_n(s) f(s) ds \right\}^2,$$

and hence that $\gamma(s, t)$ is of positive type in the square.

We observe that, from the hypotheses stated

$$f((x), s) = \sum_{n=1} x_n \theta_n(s)$$

is a linear function associated with the class F , (x) being any real point of Hilbertian space.

§ 8. Now let Ω, Ω' be two linear transformations which form a symmetric couple. It follows from the property of $f((x), s)$ stated at the end of the preceding paragraph that the integral

$$\int_a^b \Omega_s [f((x), s)] \cdot \Omega'_s [f((x), s)] ds, \quad (10)$$

exists at each point (x) . Further, if $f_r((x), s)$ is employed to denote

$\sum_{n=1}^r x_n \theta_n(s)$, we have

$$\int_a^b \Omega [f_r((x), s)] \cdot \Omega' [f_r((x), s)] ds = \sum_{m=1}^r \sum_{n=1}^r a_{mn} x_m x_n, \quad (11)$$

where
$$a_{mn} = \int_a^b \Omega [\theta_m(s)] \cdot \Omega' [\theta_n(s)] ds \quad \text{and} \quad a_{mn} = a_{nm},$$

because Ω and Ω' form a symmetric couple. Thus, recalling that the left-hand member of (11) tends to (10) as r tends to infinity,* we see that $\sum_{m,n} a_{mn} x_m x_n$ is a limited quadratic form,† which at each point (x) is represented by (10).

The form $\sum_{m,n} a_{mn} x_m x_n$ may be shown to be both limited and completely continuous. For we have

$$\begin{aligned} a_{mn}^2 &= \left\{ \int_a^b \Omega [\theta_m(s)] \cdot \Omega' [\theta_n(s)] ds \right\}^2 \\ &\leq \int_a^b \{ \Omega [\theta_m(s)] \}^2 ds \cdot \int_a^b \{ \Omega' [\theta_n(s)] \}^2 ds, \end{aligned}$$

whence, using the bounded property of a linear transformation, we have

$$a_{mn}^2 \leq M^2 M'^2 \int_a^b \{ \theta_m(s) \}^2 ds \cdot \int_a^b \{ \theta_n(s) \}^2 ds,$$

where the numbers M, M' are independent of m and n . Since the series

$$\sum_{n=1}^{\infty} \int_a^b \{ \theta_n(s) \}^2 ds,$$

is convergent by hypothesis, the double series $\sum_{m,n} a_{mn}^2$ is convergent. The convergence of the latter series is known to ensure that the quadratic form $\sum_{m,n} a_{mn} x_m x_n$ is limited and completely continuous.

§ 9. The form $\sum_{m,n} a_{mn} x_m x_n$ being limited and completely continuous, it follows that there exists an orthogonal substitution,

$$x_m' = l_{m1} x_1 + l_{m2} x_2 + \dots + l_{mn} x_n + \dots (m = 1, 2, \dots),$$

such that

$$\sum_{m,n} a_{mn} x_m x_n = \sum_{n=1}^{\infty} k_n x_n'^2, \quad (12)$$

where $k_1, k_2, \dots, k_n$ are constants, some of which may be zero. It will be recalled that the coefficients l of the substitution satisfy the relations

$$\sum_{r=1}^{\infty} l_{mr} l_{nr} = \sum_{r=1}^{\infty} l_{rm} l_{rn} = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n, \end{cases} \quad (13)$$

and that the co-ordinates of the point (x) are expressible in terms of those of the point (x') by the equations

$$x_m = l_{1m} x_1' + l_{2m} x_2' + \dots + l_{nm} x_n' + \dots (m = 1, 2, \dots).$$

* Vide § 5.

† Cf. Hellinger and Toeplitz, 'Math. Annalen,' vol. 69, pp. 326-7.

Employing the equations just written, and recalling that $\sum_{n=1} \{\theta_n(s)\}^2$ converges almost everywhere in (a, b) , it will be found that, for almost all values of s in (a, b) ,

$$f((x), s) = \sum_{n=1} x_n' \chi_n(s), \quad (14)$$

where $\chi_n(s) = \sum_{m=1} l_{nm} \theta_m(s)$. It is easily seen that $\chi_n(s)$ is a function which belongs to F, and that the series

$$\int_a^b \{\chi_1(s)\}^2 ds + \int_a^b \{\chi_2(s)\}^2 ds + \dots + \int_a^b \{\chi_n(s)\}^2 ds + \dots,$$

is convergent.* We conclude that

$$g((x')s) = x_1' \chi_1(s) + x_2' \chi_2(s) + \dots + x_n' \chi_n(s) + \dots,$$

is a linear function associated with the class F, (x') being a variable point of Hilbertian space.

Recalling that the left-hand member of (12) is (10), we see from (14) that, at each point (x') ,

$$\int_a^b \Omega[g((x'), s)] \cdot \Omega'[g((x'), s)] ds = \sum_{n=1} l_n x_n'^2. \quad (15)$$

The left-hand member of this equation can be shown, by an argument similar to that employed in § 8, to represent the quadratic form $\sum_{m,n} b_{mn} x_m' x_n'$, where

$b_{mn} = \int_a^b \Omega[\chi_m(s)] \cdot \Omega'[\chi_n(s)] ds$, and we have $b_{mn} = b_{nm}$ in virtue of the hypothesis that Ω and Ω' form a symmetric couple. Comparing the coefficients of like terms on the two sides of (15) we find that

$$\int_a^b \Omega[\chi_m(s)] \cdot \Omega'[\chi_n(s)] ds = \begin{cases} l_m & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases} \quad (16)$$

§ 10. We have seen that the convergence of (9) ensures that $\sum_{n=1} \{\theta_n(s)\}^2$ is convergent almost everywhere. Let us now suppose that (s, t) is a point in the square $a \leq s \leq b, a \leq t \leq b$ such that the series $\sum_{n=1} \{\theta_n(s)\}^2, \sum_{n=1} \{\theta_n(t)\}^2$ both converge.† Then the series $\sum_{n=1} \theta_n(s) \theta_n(t)$ is absolutely convergent and, making

* The sum is equal to that of the series $\sum_{n=1} \int_a^b \{\theta_n(s)\}^2 ds$.

† One, or more, of the functions $\theta_n(s), \theta_n(t), (n = 1, 2, \dots)$ may be multiple valued at the point under consideration. When this is the case, it must be understood that a function defined in terms of the θ 's (e.g., $\gamma(s, t)$ or one of the functions $u_n(s), u_n(t)$ below) is supposed to have the value corresponding to a particular set of determinations $\theta_n(s), \theta_n(t), (n = 1, 2, \dots)$ at the point. In the pages which follow, when any statement involving several such functions is made, it is understood that each function has the value corresponding to the *same* set of determinations.

use of (13), it will be found that

$$\gamma(s, t) = \sum_{n=1} \theta_n(s) \cdot \theta_n(t) = \sum_{n=1} \chi_n(s) \cdot \chi_n(t), \quad (17)$$

where $\chi_n(s)$, $\chi_n(t)$ have finite values for all values of n .

The functions $\chi(s)$ may be divided into two classes according as the corresponding number k is not, or is, zero. Let $u_1(s), u_2(s), \dots, u_n(s), \dots$ be the functions $\chi(s)/|k|^{\frac{1}{2}}$ for which the corresponding k is not zero; and let $U_1(s), U_2(s), \dots, U_n(s), \dots$ be the functions $\chi(s)$ for which the corresponding k is zero. Further, let μ_n be the reciprocal of the modulus of that k which corresponds to $u_n(s)$; and let δ_n be $+1$ or -1 according as the k referred to is positive or negative. Then from (16) and (17) it will be seen that we have the following theorem:—

Let $\theta_1(s), \theta_2(s), \dots, \theta_n(s), \dots$ be a countable set of functions, each of which has its square summable in (a, b) ; let the series

$$\int_a^b \{\theta_1(s)\}^2 ds + \int_a^b \{\theta_2(s)\}^2 ds + \dots + \int_a^b \{\theta_n(s)\}^2 ds + \dots, \quad (8)$$

be convergent; and let $\gamma(s, t)$ be the function of positive type which is defined in the square $a \leq s \leq b, a \leq t \leq b$ by

$$\gamma(s, t) = \sum_{n=1} \theta_n(s) \cdot \theta_n(t).$$

Further, let Ω, Ω' be a symmetric couple of linear transformations for the interval (a, b) . Then, at any point (s, t) of the square such that $\gamma(s, s)$ and $\gamma(t, t)$ are finite,* we have

$$\gamma(s, t) = \sum_{n=1} \frac{u_n(s) u_n(t)}{\mu_n} + \sum_{n=1} U_n(s) \cdot U_n(t), \quad (18)$$

where the functions $u_n(s), U_n(s)$ ($n = 1, 2, \dots$) have summable squares in (a, b) , the numbers μ_n ($n = 1, 2, \dots$) are positive, and each of the series on the right is absolutely convergent. Further, the functions $u(s), U(s)$ are such that (i)

$$\int_a^b \Omega[u_m(s)] \cdot \Omega'[u_n(s)] ds = \begin{cases} \delta_m & \text{for } n = m, \\ 0 & \text{for } n \neq m, \end{cases}$$

where δ_m is either $+1$ or -1 , and (ii) each of the pairs of functions $\{\Omega[u_m(s)], \Omega'[U_n(s)]\}$, $\{\Omega[U_m(s)], \Omega'[u_n(s)]\}$, $\{\Omega[U_m(s)], \Omega'[U_n(s)]\}$ is orthogonal for the interval (a, b) , whatever be the values of m and n .

In the theorem, we may suppose $\Omega' = \Omega$. Then $\Omega[U_n(s)]$ is orthogonal to itself, and is therefore a null function for the interval (a, b) ; further, it will be seen that δ_m is always $+1$, since (10) is, in this case, a positive quadratic form. We have thus the following corollary:—

* Vide footnote, § 10.

Let Ω be any linear transformation, and let $\gamma(s, t)$ satisfy the hypotheses of the above theorem. Then, under the same conditions as to $\gamma(s, s)$ and $\gamma(t, t)$, we have (18) where $\Omega[u_n(s)]$ ($n = 1, 2, \dots$) is an orthogonal system of functions for the interval (a, b) , and the functions $\Omega[U_n(s)]$ ($n = 1, 2, \dots$) are null functions.

By taking $\Omega = E$, where E is the unit transformation, and substituting $u_n(s) = \psi_n(s)$, $U_n(s) = \omega_n(s)$, $\mu_n = \lambda_n$, we obtain the expansion

$$\gamma(s, t) = \sum_{n=1} \frac{\psi_n(s) \psi_n(t)}{\lambda_n} + \sum_{n=1} \omega_n(s) \omega_n(t). \quad (19)$$

The functions $\psi_n(s)$ ($n = 1, 2, \dots$) form an orthogonal system of functions for the interval (a, b) , and $\omega_n(s)$ ($n = 1, 2, \dots$) are null functions. It should be recalled that $\gamma(s, t)$ is not necessarily bounded, and that (19) is valid when $\gamma(s, s)$ and $\gamma(t, t)$ are finite. There will be no difficulty in proving that

$$\psi_n(s) = \lambda_n \int_a^b \gamma(s, t) \psi_n(t) dt,$$

for any value of s such that $\gamma(s, s)$ is finite. Finally, in view of what is said in §§ 14, 15 below, it will be seen that the Fredholm theory can be extended to the function $\gamma(s, t)$.

§ 11. We may employ the corollary of the preceding paragraph to further analyse the expansion of $\gamma(s, t)$. For the function $\sum_{n=1} U_n(s) \cdot U_n(t)$ satisfies the requirements of the corollary.* It follows that we have

$$\sum_{n=1} U_n(s) U_n(t) = \sum_{n=1} \frac{v_n(s) v_n(t)}{v_n} + \sum_{n=1} V_n(s) V_n(t), \quad (20)$$

where $\Omega'[v_n(s)]$ ($n = 1, 2, \dots$) form an orthogonal system of functions for (a, b) , v_n ($n = 1, 2, \dots$) are positive numbers, and $\Omega'[V_n(s)]$ ($n = 1, 2, \dots$) are null functions. As the functions $v_n(s)$, $V_n(s)$ are linear functions of $U_n(s)$ ($n = 1, 2, \dots$), it will be clear that the orthogonal properties stated under (ii) in the theorem of the preceding paragraph will hold when the function symbol, U , is replaced by either v or V . There is, however, no need to state the orthogonal properties which involve functions $\Omega'[V_n(s)]$, since these are a consequence of the fact that $\Omega'[V_n(s)]$ ($n = 1, 2, \dots$) are null functions.

Next, we can apply the same corollary to $\sum V_n(s) V_n(t)$ and so obtain

$$\sum_{n=1} V_n(s) V_n(t) = \sum_{n=1} \frac{w_n(s) \cdot w_n(t)}{\rho_n} + \sum_{n=1} W_n(s) W_n(t), \quad (21)$$

where $\Omega[w_n(s)]$ ($n = 1, 2, \dots$) form an orthogonal system of functions for (a, b) , ρ_n ($n = 1, 2, \dots$) are positive numbers, and $\Omega[W_n(s)]$ ($n = 1, 2, \dots$) are

* The hypothesis that $\gamma(s, s)$ and $\gamma(t, t)$ are finite is still understood.

null functions. After what was said above, the reader will have no difficulty in perceiving that the orthogonal properties are as stated in the theorem which follows.

Employing (20) and (21), it will be seen that we have the following theorem :—

Under the hypotheses of the theorem of the last paragraph, we have

$$\gamma(s, t) = \sum_{n=1} \frac{u_n(s) u_n(t)}{\mu_n} + \sum_{n=1} \frac{v_n(s) v_n(t)}{\nu_n} + \sum_{n=1} \frac{w_n(s) w_n(t)}{\rho_n} + \sum_{n=1} W_n(s) \cdot W_n(t), \quad (22)$$

where the functions $u(s)$, $v(s)$, $w(s)$, $W(s)$ have summable squares in (a, b) , the numbers μ , ν , ρ are positive, and each of the series on the right is absolutely convergent. Further, the functions $u(s)$, $v(s)$, $w(s)$, $W(s)$ are such that (i)

$$\int_a^b \Omega[u_m(s)] \cdot \Omega'[u_n(s)] ds = \begin{cases} \delta_m & \text{for } n = m, \\ 0 & \text{for } n \neq m, \end{cases}$$

where δ_m is either $+1$ or -1 ; (ii) the functions $\Omega'[v_n(s)]$ ($n = 1, 2, \dots$) and the functions $\Omega[w_n(s)]$ ($n = 1, 2, \dots$) each constitute an orthogonal system of functions for the interval (a, b) ; (iii) each of the pairs of functions $\{\Omega[u_m(s)], \Omega'[v_n(s)]\}$, $\{\Omega[v_m(s)], \Omega'[u_n(s)]\}$, $\{\Omega[v_m(s)], \Omega'[v_n(s)]\}$, $\{\Omega[w_m(s)], \Omega'[u_n(s)]\}$, $\{\Omega[w_m(s)], \Omega'[v_n(s)]\}$, is orthogonal for (a, b) , whatever be the values of m and n ; and (iv) $\Omega'[w_n(s)]$, $\Omega[W_n(s)]$, $\Omega'[W_n(s)]$ are null functions.

The function $\gamma(s, s)$ has a single finite value almost everywhere in (a, b) . It follows that (22) is valid almost everywhere with respect to s , and almost everywhere with respect to t .

It is perhaps unnecessary to point out that there is a second expansion of $\gamma(s, t)$ of the form (22); for this expansion, the statements of the above theorem are applicable, provided that Ω and Ω' are everywhere interchanged.* In either case, one or more of the series on the right of (22) may contain only a finite number of terms, and it is possible that one or more of the series may be non-existent.

§ 12. Referring to the theorem of § 11, let us denote the sum function of the series $\sum_{n=1} u_n(s) u_n(t) / \mu_n$ by $\gamma_1(s, t)$. Since $\gamma(s, s)$ is summable in (a, b) , it

will be clear that $\sum_{n=1} \int_a^b \{u_n(s) / \mu_n^{\frac{1}{2}}\}^2 ds$ is convergent; and hence that

* To derive this expansion Ω and Ω' must, of course, be interchanged in the argument at the commencement of the paragraph.

$\sum_{n=1} x_n u_n(s)/\mu_n^{\frac{1}{2}}$ is a linear function associated with F. It follows from what was said in § 5, that

$$\Omega_s \left[\sum_{n=1} x_n u_n(s)/\mu_n^{\frac{1}{2}} \right] = \sum_{n=1} x_n \Omega [u_n(s)]/\mu_n^{\frac{1}{2}}$$

almost everywhere in (a, b) . Observing that $(u(t)/\mu^{\frac{1}{2}})$ is a point of Hilbertian space for almost all values of t in the interval (a, b) , we see that

$$\Omega_s [\gamma_1(s, t)] = \sum_{n=1} \Omega [u_n(s)] u_n(t)/\mu_n, \quad (23)$$

almost everywhere with respect to s , and almost everywhere with respect to t .

Again we have

$$\Omega_t' \left[\sum_{n=1} x_n u_n(t)/\mu_n^{\frac{1}{2}} \right] = \sum_{n=1} x_n \Omega_t' [u_n(t)]/\mu_n^{\frac{1}{2}}$$

almost everywhere in (a, b) and $(\Omega [u(s)]/\mu^{\frac{1}{2}})$ is a point of Hilbertian space for almost all values of s in the interval $a \leq s \leq b$. It follows that we have

$$\Omega_t' \left[\sum_{n=1} \Omega [u_n(s)] \cdot u_n(t)/\mu_n \right] = \sum_{n=1} \Omega [u_n(s)] \cdot \Omega' [u_n(t)]/\mu_n$$

almost everywhere with respect to s , and almost everywhere with respect to t . We conclude from (23) that

$$\Omega_t' [\Omega_s [\gamma_1(s, t)]] = \sum_{n=1} \Omega [u_n(s)] \Omega' [u_n(t)]/\mu_n, \quad (24)$$

almost everywhere with respect to s , and almost everywhere with respect to t .

It will be clear that in a similar way we can prove the validity of

$$\Omega_s [\Omega_t' [\gamma_1(s, t)]] = \sum_{n=1} \Omega [u_n(s)] \cdot \Omega' [u_n(t)]/\mu_n, \quad (25)$$

under similar circumstances. Adopting a slightly abbreviated mode of expression for the left-hand members of (24) and (25), we see that

$$\Omega_s \Omega_t' [\gamma_1(s, t)] = \Omega_t' \Omega_s [\gamma_1(s, t)] = \sum_{n=1} \Omega [u_n(s)] \cdot \Omega' [u_n(t)]/\mu_n$$

almost everywhere with respect to s , and almost everywhere with respect to t .

Similar results may be established in respect to the other series on the right of (22), and hence in respect to $\gamma(s, t)$. Further, it will be clear that each series such as that on the right of (25) is absolutely convergent almost everywhere with respect to s , and almost everywhere with respect to t .

§ 13. Let us define a function $\kappa(s, t)$ by the rule that it has the common value of $\Omega_s \Omega_t' [\gamma(s, t)]$ and $\Omega_t' \Omega_s [\gamma(s, t)]$ at points where these are equal. Further, let

$$\begin{aligned} \phi_n(s) &= \delta_n \cdot \Omega [u_n(s)], & \psi_n(s) &= \Omega' [u_n(s)], \\ \xi_n(s) &= \Omega [v_n(s)], & \eta_n(s) &= \Omega' [v_n(s)], \\ \zeta_n(s) &= \Omega [w_n(s)], & \omega_n^{(1)}(s) &= \Omega' [w_n(s)], \\ \omega_n^{(2)}(s) &= \Omega [W_n(s)], & \omega_n^{(3)}(s) &= \Omega' [W_n(s)], \end{aligned}$$

and let $\lambda_n = \delta_n \mu_n$. It will be seen that we have

$$\begin{aligned} \kappa(s, t) = \sum_{n=1} \frac{\phi_n(s) \psi_n(t)}{\lambda_n} + \sum_{n=1} \frac{\xi_n(s) \eta_n(t)}{\nu_n} \\ + \sum_{n=1} \frac{\xi_n(s) \omega_n^{(1)}(t)}{\rho_n} + \sum_{n=1} \omega_n^{(2)}(s) \omega_n^{(3)}(t),^* \end{aligned} \quad (26)$$

almost everywhere with respect to s , and almost everywhere with respect to t ; also it can be established that term by term integration with respect to either variable of the series on the right is justified. From these facts and the properties of the functions $\phi_n(s)$, $\psi_n(s)$, ..., which follow from the statements (i)–(iv) in the theorem of § 11, we find that, almost everywhere in (a, b) , we have

$$\left. \begin{aligned} \phi_n(s) &= \lambda_n \int_a^b \kappa(s, t) \phi_n(t) dt \\ \psi_n(t) &= \lambda_n \int_a^b \psi_n(s) \kappa(s, t) ds \end{aligned} \right\} \quad (27)$$

We can now state the following theorem:—

Let $\gamma(s, t)$ be a function of positive type defined as in the theorem of § 10; let Ω, Ω' be a symmetric couple of linear transformations; and let $\kappa(s, t)$ be the common value of $\Omega_s \Omega'_t [\gamma(s, t)]$ and $\Omega'_t \Omega_s [\gamma(s, t)]$ at points where these are equal. Then the equation (26) is valid, and the right-hand member is absolutely convergent, almost everywhere with respect to s , and also everywhere with respect to t . The functions

$$\begin{aligned} \phi_1(s), \phi_2(s), \dots, \phi_n(s), \dots, \\ \psi_1(s), \psi_2(s), \dots, \psi_n(s), \dots, \end{aligned}$$

form a bi-orthogonal system of functions for the interval (a, b) , and each of the homogeneous equations (27) is satisfied almost everywhere in the interval. The symbols ν_n, ρ_n ($n = 1, 2, \dots$) denote positive constants. The functions $\eta_n(s)$ ($n = 1, 2, \dots$) and the functions $\xi_n(s)$ ($n = 1, 2, \dots$) each constitute an orthogonal system of functions for the interval (a, b) . Each of the pairs of functions $\{\phi_m(s), \eta_n(s)\}$, $\{\xi_m(s), \psi_n(s)\}$, $\{\xi_m(s), \eta_n(s)\}$, $\{\xi_m(s), \psi_n(s)\}$, $\{\xi_m(s), \eta_n(s)\}$, is orthogonal for the interval (a, b) , whatever be the values of m and n . Lastly, for all values of n the functions $\omega_n^{(1)}(s)$, $\omega_n^{(2)}(s)$, $\omega_n^{(3)}(s)$ are null functions.

In view of a remark at the end of § 11, it will be seen that there is a second expansion for $\kappa(s, t)$, similar to the above.

* For the case under consideration, the series $\sum_{n=1} \omega_n^{(2)}(s) \omega_n^{(3)}(t)$ may be omitted, since its sum function is zero almost everywhere with respect to s , and almost everywhere with respect to t . The series is retained here in view of the expansion of $\gamma(s, t)$, and because it cannot be ignored when the transformations are precisely distributive in Hilbertian space (*vide* § 17).

§ 14. It seems desirable to make some observations in regard to the theorem of the last paragraph.

In the first place, it will be clear that the Fredholm theory, as usually developed, is not applicable to $\kappa(s, t)$, since this function may be unbounded in such a general manner. We may, however, obtain results similar to those of the theory referred to, as follows:—

If we define $K(\lambda; s, t)$ to be the sum function of the series on the right of (26) after λ_n has been replaced by $\lambda_n - \lambda$ in the first component series, we find that the square of $K(\lambda; s, t)$ is summable in $a \leq s \leq b$, $a \leq t \leq b$, provided that λ has not one of the values $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$.

It is easily verified that

$$K(\lambda; s, t) - \kappa(s, t) = \lambda \int_a^b K(\lambda; s, u) \kappa(u, t) du = \lambda \int_a^b \kappa(s, u) K(\lambda; u, t) du, \quad (28)$$

almost everywhere with respect to s , and almost everywhere with respect to t ; and hence that, if $f(s)$ belongs to the class F, the equation

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t) g(t) dt, \quad (29)$$

is satisfied almost everywhere in (a, b) by

$$g(s) = f(s) + \lambda \int_a^b K(\lambda; s, t) f(t) dt,$$

provided that λ is not one of the numbers λ_n . It will be clear that a corresponding remark applies in regard to the solution of the equation adjoint to (29).

The numbers $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ may be called *singular values* of $\kappa(s, t)$.*

§ 15. Referring to §§ 8, 9, it will be found that

$$|k_n| = \left| \int_a^b \Omega[\chi_n(s)] \cdot \Omega'[\chi_n(s)] ds \right| \leq MM' \int_a^b \{\chi_n(s)\}^2 ds.$$

Since $\sum_{n=1}^{\infty} \{\chi_n(s)\}^2 = \sum_{n=1}^{\infty} \{\theta_n(s)\}^2$ almost everywhere, we see that

$$\sum_{n=1}^m |k_n| \leq MM' \int_a^b \sum_{n=1}^m \{\theta_n(s)\}^2 ds.$$

* The reader will observe that by a known property of completely continuous quadratic forms, $\lim_{n \rightarrow \infty} k_n = 0$, and hence that $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$, can be divided into sets, each containing a finite number of equal singular values, and such that members of distinct sets are unequal.

It follows that the series

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n} + \dots, \quad (30)$$

is absolutely convergent, and that $\sum |1/\lambda_n|$ is $\leq MM' \int_a^b \gamma(s, s) ds$.*

The series (30) being absolutely convergent, it follows that

$$D(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)$$

is an integral function of λ . Also from the expansion of $\kappa(s, t)$, it will be found that, for all integral values of r ,

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^r} = \int_a^b \kappa_r(s, s) ds,$$

where $\kappa_1(s, t) = \kappa(s, t)$ and $\kappa_{r+1}(s, t) = \int_a^b \kappa_r(s, u) \kappa(u, t) du$.

From these facts it can be shown that

$$\begin{aligned} D(\lambda) = & 1 - \lambda \int_a^b \kappa(s_1, s_1) ds_1 + \frac{\lambda^2}{2!} \int_a^b \int_a^b \kappa \begin{pmatrix} s_1 & s_2 \\ s_1 & s_2 \end{pmatrix} ds_1 ds_2 \\ & - \dots + (-)^n \frac{\lambda^n}{n!} \int_a^b \dots \int_a^b \kappa \begin{pmatrix} s_1, s_2, \dots, s_n \\ s_1, s_2, \dots, s_n \end{pmatrix} ds_1 ds_2 \dots ds_n + \dots, \end{aligned}$$

as in the usual Fredholm theory. Observing that $D(\lambda) K(\lambda; s, t)$ is an integral function of λ , and making use of (28) it can be shown that, almost everywhere with respect to s and almost everywhere with respect to t , we have

$$D(\lambda) K(\lambda; s, t) = D(\lambda; s, t),$$

where $D(\lambda; s, t)$ is the integral function of λ defined by

$$\kappa(s, t) - \lambda \int_a^b \kappa \begin{pmatrix} s & s_1 \\ t & s_1 \end{pmatrix} ds_1 + \dots + (-)^n \frac{\lambda^n}{n!} \int_a^b \dots \int_a^b \kappa \begin{pmatrix} s, s_1, s_2, \dots, s_n \\ t, s_1, s_2, \dots, s_n \end{pmatrix} ds_1 ds_2 \dots ds_n + \dots$$

The function $K(\lambda; s, t)$ is thus expressed as the quotient of two integral functions of λ , for almost all values of s , and almost all values of t .

Further developments in this direction may be left to the reader.

§ 16. It can be proved that, if the equation

$$f(s) = \lambda \int_a^b \kappa(s, t) f(t) dt$$

* This is a generalisation of a theorem given by Garbe ('Math. Annalen,' vol. 76, pp. 545-6) for Hilbert's function $a(s) \gamma(st)$, where $a(s) = +1$ in certain sub-intervals of (a, b) , and $= -1$ in the complementary set. The absolute convergence may also be inferred from the fact (*vide* below) that $\sum 1/\lambda_n = \int_a^b \kappa(s, s) ds$, and that the order of the singular values is arbitrary.

is satisfied almost everywhere in (a, b) by a function $f(s)$, which is not a null-function, then λ must be one of the singular values λ_n ; also $f(s)$ must, almost everywhere in (a, b) , be a linear combination of the functions $\phi_n(s)$ corresponding to the singular value.

Again, if $\Gamma(s, t) = \Omega_s' \Omega_t' [\gamma(s, t)]$, it will be found that

$$\int_a^b \Gamma(s, u) \kappa(u, t) du = \sum_{n=1} \frac{\psi_n(s) \psi_n(t)}{\lambda_n \mu_n},$$

almost everywhere with respect to s , and almost everywhere with respect to t ; moreover $\int_a^b \Gamma(s, u) \phi_n(u) du$ being almost everywhere equal to $\psi_n(s)/\mu_n$ is never a null-function. Giving an extended meaning to the term as previously used,* we may, therefore, say that $\kappa(s, t)$ is completely symmetrisable on the left by $\Gamma(s, t)$. In a similar sense, $\kappa(s, t)$ is completely symmetrisable on the right by $\Gamma'(s, t) = \Omega_s \Omega_t [\gamma(s, t)]$.

Finally, if the equation (26) is multiplied through by $f(s) g(t)$, two functions which belong to F, we find after integration that

$$\begin{aligned} \int_a^b \int_a^b \kappa(s, t) f(s) g(t) ds dt = & \sum_{n=1} \frac{\int_a^b \phi_n(s) f(s) ds \cdot \int_a^b \psi_n(t) g(t) dt}{\lambda_n} \\ & + \sum_{n=1} \frac{\int_a^b \xi_n(s) f(s) ds \cdot \int_a^b \eta_n(t) g(t) dt}{\mu_n}. \quad (31) \end{aligned}$$

This result is, of course, analogous to Hilbert's expansion of

$$\int_a^b \int_a^b \kappa(s, t) f(s) g(t) ds dt,$$

for the case when $\kappa(s, t)$ is symmetric.

§ 17. Let $\gamma(s, t)$ be a function which is bounded, summable, and of positive type in the square $a \leq s \leq b$, $a \leq t \leq b$; further, let the function be continuous with respect to s in the interval (a, b) for each fixed value of t in this interval. The fact that $\gamma(s, t)$ is symmetric permits the interchange of s and t in the last statement. Then it is known that†

$$\gamma(s, t) = \sum_{n=1} \frac{\pi_n(s) \cdot \pi_n(t)}{\sigma_n},$$

where $\pi_n(s)$ ($n = 1, 2, \dots$) constitute a complete system of fundamental functions of $\gamma(s, t)$, and σ_n is the (positive) singular value corresponding to

* Cf. 'Roy. Soc. Proc.' A, vol. 97, p. 402. The continuity properties assumed in this paper excluded the case of a null-function which is not zero everywhere.

† Vide 'Roy. Soc. Proc.' A, vol. 97, p. 407.

$\pi_n(s)$. The functions $\pi_n(s)$ are continuous in (a, b) and the series $\sum_{n=1} \{\pi_n(s)\}^2 / \sigma_n$ is convergent for each value of s which is $\geq a$ and $\leq b$. It follows that, if we write $\pi_n(s) / \sigma_n^{\frac{1}{2}} = \theta_n(s)$, we have

$$\gamma(s, t) = \sum_{n=1} \theta_n(s) \theta_n(t),$$

where the functions $\theta_n(s)$ satisfy the requirements of the theorem of § 10. The expansion (22) is thus seen to be applicable at *any* point of the square $a \leq s \leq b, a \leq t \leq b$.

Let us now suppose that the transformations Ω, Ω' are well defined and precisely distributive in Hilbertian space, and that, taken together, they constitute a symmetric couple. Proceeding by an argument parallel to that of § 12, it will be found that we have

$$\Omega_s \Omega'_t [\gamma(s, t)] = \Omega'_t \Omega_s [\gamma(s, t)]$$

at *all* points; further, if the common value is denoted by $\kappa(s, t)$, the expansion (26) is valid at *all* points of the square $a \leq s \leq b, a \leq t \leq b$.

There can be no difficulty in developing a theory of the integral equation with $\kappa(s, t)$ as kernel on the lines sketched in §§ 14, 15. The reader will observe the essential difference that statements, which in the paragraphs referred to are true almost everywhere, are now true at *all* points.

The functions $\theta_n(s) = \pi_n(s) / \sigma_n^{\frac{1}{2}}$ being all continuous in (a, b) , it follows that the functions $u(s), v(s), w(s), W(s)$ are all continuous in the interval. Hence, if the transformation Ω' is perfect in an interval (a', b') , where $a - a' \leq b' - b$, the functions $\psi_n(s), \eta_n(s), \omega_n^{(1)}(s), \omega^{(3)}(s)$ are all continuous in (a', b') . Since a continuous null-function is zero everywhere, we see that $\omega_n^{(1)}(s), \omega_n^{(3)}(s)$ are evanescent.

§ 18. We may now state the following expansion theorem:—

Let $\gamma(s, t)$ be a function which is bounded, summable, and of positive type, in the square $a \leq s \leq b, a \leq t \leq b$; further, let the function be continuous with respect to s in the interval (a, b) for each fixed value of t in the same interval. Finally, let Ω and Ω' be two linear transformations which are precisely distributive in Hilbertian space, and which together constitute a symmetric couple. Then, at all points of the square $a \leq s \leq b, a \leq t \leq b$, we have $\Omega_s \Omega'_t [\gamma(s, t)] = \Omega'_t \Omega_s [\gamma(s, t)]$, and, denoting the common value of these functions by $\kappa(s, t)$, we have the expansion (26), wherein each of the series on the right is absolutely convergent. The functions

$$\phi_1(s), \phi_2(s), \dots, \phi_n(s), \dots$$

$$\psi_1(s), \psi_2(s), \dots, \psi_n(s), \dots$$

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which occur in the expansion form a bi-orthogonal system of functions for the interval (a, b) ; and $\phi_n(s), \psi_n(s)$ satisfy the equations

$$\begin{aligned}\phi_n(s) &= \lambda_n \int_a^b \kappa(s, t) \phi_n(t) dt, \\ \psi_n(t) &= \lambda_n \int_a^b \psi_n(s) \kappa(s, t) ds,\end{aligned}$$

at all points of the interval (a, b) . The remaining functions which occur in the expansion have the properties stated in the theorem of § 13.

Moreover, if Ω' is perfect in an interval (a', b') contained in (a, b) , the expansion takes the form

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s) \psi_n(t)}{\lambda_n} + \sum_{n=1} \frac{\xi_n(s) \eta_n(t)}{\nu_n}, \quad (32)$$

for $a \leq s \leq b, a' \leq t \leq b'$; and the functions $\psi_n(s), \eta_n(s)$ are continuous in (a', b') . The corresponding statement, which applies when Ω is perfect in a sub-interval of (a, b) , will be obvious.

The reader will recall that, in any given case, there is always a second expansion for $\kappa(s, t)$ (vide §§ 11, 13).

§ 19. In conclusion, it should be observed that conditions may be stated under which the expansions given in §§ 13, 18 assume a special character. The conditions are generalisations of those stated at the end of § 9 of the communication of April 12 for the case of the function of $\alpha(s)\gamma(s, t)$.

For example, let the hypotheses of the theorem of § 13 be adopted, and let $\Gamma(s, t) = \Omega_s' \Omega_t' [\gamma(s, t)]$ be definite. Employing the equation (31) of § 16, it is easily shown that $\xi_n(s)$ ($n = 1, 2, \dots$) are null functions, and that $\zeta_n(s)$ ($n = 1, 2, \dots$) cannot exist. It follows that the expansion (26) takes the simpler form

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s) \psi_n(t)}{\lambda_n} + \sum_{n=1} \frac{\omega_n^{(1)}(s) \eta_n(t)}{\nu_n} + \sum_{n=1} \omega_n^{(2)}(s) \cdot \omega_n^{(3)}(t),$$

where the functions which occur have the properties stated in the theorem of § 13. This expansion will hold at all points of $a \leq s \leq b, a \leq t \leq b$, when $\gamma(s, t)$ satisfies the requirements of the theorem of the preceding paragraph and, as there, Ω, Ω' are precisely distributive in Hilbertian space. If we further suppose that Ω is perfect in an interval (a', b') , $\omega_n^{(1)}(s), \omega_n^{(2)}(s)$ are continuous null functions, and therefore vanish everywhere in the interval. We thus have the expansion

$$\kappa(s, t) = \sum_{n=1} \frac{\phi_n(s) \psi_n(t)}{\lambda_n},$$

which is valid when s belongs to (a', b') and t to (a, b) .

It should be observed that the hypothesis that $\gamma(s, t)$ is definite does not in itself ensure that $\Gamma(s, t)$ is definite. This, however, will certainly be the case when Ω' has an inverse.